



Estimating Volatilities and Correlations

Chapter 21

Standard Approach to Estimating Volatility (page 477)

- Define σ_n as the volatility per day between day $n-1$ and day n , as estimated at end of day $n-1$
- Define S_i as the value of market variable at end of day i
- Define $u_i = \ln(S_i/S_{i-1})$

$$\sigma_n^2 = \frac{1}{m-1} \sum_{i=1}^m (u_{n-i} - \bar{u})^2$$

$$\bar{u} = \frac{1}{m} \sum_{i=1}^m u_{n-i}$$

Simplifications Usually Made (page 478)

- Define u_i as $(S_i - S_{i-1})/S_{i-1}$
- Assume that the mean value of u_i is zero
- Replace $m-1$ by m

This gives

$$\sigma_n^2 = \frac{1}{m} \sum_{i=1}^m u_{n-i}^2$$

Weighting Scheme

Instead of assigning equal weights to the observations we can set

$$\sigma_n^2 = \sum_{i=1}^m \alpha_i u_{n-i}^2$$

where

$$\sum_{i=1}^m \alpha_i = 1$$

ARCH(m) Model (page 479)

In an ARCH(m) model we also assign some weight to the long-run variance rate, V_L :

$$\sigma_n^2 = \gamma V_L + \sum_{i=1}^m \alpha_i u_{n-i}^2$$

where

$$\gamma + \sum_{i=1}^m \alpha_i = 1$$

EWMA Model

- In an exponentially weighted moving average model, the weights assigned to the u^2 decline exponentially as we move back through time
- This leads to

$$\sigma_n^2 = \lambda \sigma_{n-1}^2 + (1 - \lambda) u_{n-1}^2$$

Attractions of EWMA

- Relatively little data needs to be stored
- We need only remember the current estimate of the variance rate and the most recent observation on the market variable
- Tracks volatility changes
- RiskMetrics uses $\lambda = 0.94$ for daily volatility forecasting

GARCH (1,1) page 481

In GARCH (1,1) we assign some weight to the long-run average variance rate

$$\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

Since weights must sum to 1

$$\gamma + \alpha + \beta = 1$$

GARCH (1,1) continued

Setting $\omega = \gamma V$ the GARCH (1,1) model is

$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

and

$$V_L = \frac{\omega}{1 - \alpha - \beta}$$

Example (Example 21.2, page 481)

- Suppose

$$\sigma_n^2 = 0.0000002 + 0.13u_{n-1}^2 + 0.86\sigma_{n-1}^2$$

- The long-run variance rate is 0.0002 so that the long-run volatility per day is 1.4%

Example continued

- Suppose that the current estimate of the volatility is 1.6% per day and the most recent percentage change in the market variable is 1%.
- The new variance rate is

$$0.000002 + 0.13 \times 0.0001 + 0.86 \times 0.000256 = 0.00023336$$

The new volatility is 1.53% per day

GARCH (p,q)

$$\sigma_n^2 = \omega + \sum_{i=1}^p \alpha_i u_{n-i}^2 + \sum_{j=1}^q \beta_j \sigma_{n-j}^2$$

Maximum Likelihood Methods

- In maximum likelihood methods we choose parameters that maximize the likelihood of the observations occurring

Example 1

- We observe that a certain event happens one time in ten trials. What is our estimate of the proportion of the time, p , that it happens?
- The probability of the event happening on one particular trial and not on the other $p(1-p)^9$ is
- We maximize this to obtain a maximum likelihood estimate. Result: $p=0.1$

Example 2

Estimate the variance of observations from a normal distribution with mean zero

Maximize
$$\prod_{i=1}^m \left[\frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{u_i^2}{2v}\right) \right]$$

Taking logarithm this is equivalent to maximizing

$$\sum_{i=1}^m \left[-\ln(v) - \frac{u_i^2}{v} \right]$$

Result:
$$v = \frac{1}{m} \sum_{i=1}^m u_i^2$$

Application to GARCH

We choose parameters that maximize

$$\prod_{i=1}^m \frac{1}{\sqrt{2\pi v_i}} \exp\left(-\frac{u_i^2}{2v_i}\right)$$

or

$$\sum_{i=1}^m \left[-\ln(v_i) - \frac{u_i^2}{v_i} \right]$$

Excel Application (Table 21.1, page 485)

- Start with trial values of ω , α , and β
- Update variances
- Calculate
$$\sum_{i=1}^m \left[-\ln(v_i) - \frac{u_i^2}{v_i} \right]$$
- Use solver to search for values of ω , α , and β that maximize this objective function
- Important note: set up spreadsheet so that you are searching for three numbers that are the same order of magnitude (See page 486)

Variance Targeting

- One way of implementing GARCH(1,1) that increases stability is by using variance targeting
- We set the long-run average volatility equal to the sample variance
- Only two other parameters then have to be estimated

How Good is the Model?

- The Ljung-Box statistic tests for autocorrelation
- We compare the autocorrelation of the u_i^2 with the autocorrelation of the u_i^2/σ_i^2

Forecasting Future Volatility

(equation 21.13, page 488)

A few lines of algebra shows that

$$E[\sigma_{n+k}^2] = V_L + (\alpha + \beta)^k (\sigma_n^2 - V_L)$$

The variance rate for an option
expiring on day m is

$$\frac{1}{m} \sum_{k=0}^{m-1} E[\sigma_{n+k}^2]$$

Forecasting Future

Volatility continued (equation 21.14, page 489)

Define

$$a = \ln \frac{1}{\alpha + \beta}$$

The volatility per annum for a T -day option is

$$\sqrt{252 \left(V_L + \frac{1 - e^{-aT}}{aT} [V(0) - V_L] \right)}$$

Volatility Term Structures (Table 21.4)

- The GARCH (1,1) suggests that, when calculating vega, we should shift the long maturity volatilities less than the short maturity volatilities
- Impact of 1% change in instantaneous volatility for Japanese yen example:

Option Life (days)	10	30	50	100	500
Volatility increase (%)	0.84	0.6 1	0.46	0.27	0.06

Correlations and Covariances (page 491-492)

Define $x_i = (X_i - X_{i-1})/X_{i-1}$ and $y_i = (Y_i - Y_{i-1})/Y_{i-1}$

Also

$\sigma_{x,n}$: daily vol of X calculated on day $n-1$

$\sigma_{y,n}$: daily vol of Y calculated on day $n-1$

cov_n : covariance calculated on day $n-1$

The correlation is $\text{cov}_n / (\sigma_{u,n} \sigma_{v,n})$

Updating Correlations

- We can use similar models to those for volatilities
- Under EWMA

$$\text{cov}_n = \lambda \text{cov}_{n-1} + (1-\lambda)x_{n-1}y_{n-1}$$

Positive Finite Definite Condition

A variance-covariance matrix, Ω , is internally consistent if the positive semi-definite condition

$$\mathbf{w}^T \Omega \mathbf{w} \geq 0$$

for all vectors $\mathbf{w}$

Example

The variance covariance matrix

$$\begin{pmatrix} 1 & 0 & 0.9 \\ 0 & 1 & 0.9 \\ 0.9 & 0.9 & 1 \end{pmatrix}$$

is not internally consistent